ELEMENTARY ROW OPERATIONS

- <u>Replace</u> any row by itself and the sum or multiple of another row
- <u>Interchange</u> - swap the order of any two rows
- <u>Scale</u> - multiply all entries in a row by a nonzero constant.

1. two matrices are row equivalent if they can be transformed into one another
2. If the augmented matrices of two linear systems are row equivalent, then those two systems have the same solution.

SECTION 1.2

- Want to get to echelon form in the augmented mtx.

$$\begin{bmatrix} \blacksquare & x & x & x \\ 0 & \blacksquare & x & x \\ 0 & 0 & \blacksquare & x \end{bmatrix}$$

$\blacksquare$ - non zero entry

x - entry that may be zero

$$\begin{bmatrix} \blacksquare & x & x & x & x \\ 0 & 0 & 0 & \blacksquare & x \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Echelon form

Inconsistent if non zero.

<u>DEFINITION</u> - ECHELON FORM

1. All non zero rows above any zero rows
2. As you move down the leading entry moves to the right.
3. All entries in a column below a leading entry are zero.

- We want to simplify Echelon Form to reduced Echelon form by removing non zeros above leading entries

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

DEFINITION - Reduced Echelon Matrices

1. Matrix is in Echelon Form
2. the leading order in each row is 1
3. the leading order should be the only non zero in the column.

$$x_1 = 7x_2 - 6x_4 + 5$$
$$x_3 = 2x_4 + 3$$

$$x_2 \begin{bmatrix} 7 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -6 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \\ 3 \\ 3 \end{bmatrix}$$

EX:

$$\begin{bmatrix} 2 & 0 & 1 & 2 \\ -1 & -1 & 2 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & -2 & 0 \\ 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -2 & 5 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & -2 & 5 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 7 & 0 \end{bmatrix}$$

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1.2 continued

$$x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ u \\ 3 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \\ 3 \end{bmatrix}$$

THEORUM

→ A linear system is consistent $\iff$ the augmented matrix does NOT have a row of of the form $[0 \dots\dots 0|b]$ for $b \neq 0$

NOTE: $[0 \dots 0|b]$ indicates that $x_n = 0$ not an inconsistency.

- 1.3 VECTOR EQUATIONS

→ Vector - matrix with one column

Ex: $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ or $\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}$, $a, b \in \mathbb{R}$

So $\vec{u}, \vec{x} \in \mathbb{R}^2$

Since $\mathbb{R}^2$ is in the standard x, y plane we have a nice geometric interpretation in $\mathbb{R}^2$

→ Adding Vectors:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-3 \\ 2+1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

→ Scalar Multiplication:

for $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$, $c\vec{u}$ for $c \in \mathbb{R}$ is $c\vec{u} = \begin{bmatrix} ca \\ cb \end{bmatrix}$

In $\mathbb{R}^3$, $\vec{x} \in \mathbb{R}^3$ where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

VECTOR PROPERTIES

1. There is a vector $\vec{0} \in \mathbb{R}^n$ w/ all entries zero
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
3. $\vec{0} + \vec{u} = \vec{u}$
4. $\vec{u} + (-\vec{u}) = \vec{0}$, $\vec{u} - \vec{u} = 0$
5. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$

LINEAR COMBINATIONS

→ Definition: Given a set of vectors $v_1, \dots, v_p$ in $\mathbb{R}^n$ and scalars $c_1, \dots, c_p$ then $\vec{y} = c_1\vec{v}_1, c_2\vec{v}_2, \dots, c_p\vec{v}_p = \sum_{i=1}^{p} c_i v_i$

is a linear combination of the vectors $\vec{v}_1, \dots \vec{v}_p$.

EX: is there a combination of vectors $\vec{u}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ Which makes $\vec{b} = \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix}$

SOLUTION: $\left[\begin{array}{cc|c} 2 & 1 & 5 \\ -1 & -1 & -2 \\ 1 & 3 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & -2 \\ 2 & 1 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & -5 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 = 3$, $x_2 = -1$

→ Definition: (Span) is the collection of all possible linear combinations

- In $\mathbb{R}^2$, span $\{\vec{u}\}$ is $c\vec{u}$
- In $\mathbb{R}^3$, Span $\{\vec{u}, \vec{v}\}$ is a plane

1.4

Definition: MATRIX VECTOR PRODUCT

- If A is an $m \times n$ mtx whose columns are $\vec{a}_1 \dots \vec{a}_n$ & $\vec{x} \in \mathbb{R}^n$ then $A\vec{x} = x_1a_1 + x_2a_2 + \dots + x_na_n$

$\rightarrow A\vec{x} = [a_1 \dots a_n]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^{n} x_ia_i$

NOTE: # of columns in A must equal number of rows in $\vec{x}$

Ex: $$\begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 3 \\ 0 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = 1\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} - 1\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

Recall: Linear systems Correspond to vector equations

THEORUM: If A is an $m \times n$ mtx such that $A\vec{x} = \vec{b}$, the solution set is $x_1a_1 + x_2a_2 + \dots + a_nx_n = \vec{b}$

Question: Does $A\vec{x} = \vec{b}$ have a solution for All $\vec{b}$?

Ex: $A = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 2 & -2 \end{bmatrix}$ $\quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ $\quad$ Does $A\vec{x} = \vec{b}$?

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & b_1 \\ 2 & 1 & 1 & b_2 \\ 3 & 2 & 2 & b_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 4 & b_1 \\ 0 & 1 & -7 & b_2 - 2b_1 \\ 0 & 2 & -14 & b_3 - 3b_1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 4 & b_1 \\ 0 & 1 & -7 & b_2 - 2b_1 \\ 0 & 0 & 0 & (b_3 - b_1) - 2(b_2 - 2b_1) \end{array}\right]$$

$\hookrightarrow b_3 + b_1 - 2b$

$\therefore$ The system will be consistent $\iff$ $b_3 + b_1 - 2b_2 = 0$

This plane $(x_1 - 2x_2 + x_3 = 0)$, must be the set of all linear combinations, i.e. the SPAN of A

THEORUM: If A is an $m \times n$ mtx then the following are equivelant

ⓐ $\forall \vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ has a unique solution

ⓑ Any $\vec{b} \in \mathbb{R}^n$ is a linear combination of the colums of A

ⓒ The columns of A span $\mathbb{R}^n$

ⓓ A has a pivot column in every row, i.e. every column is a pivot column.

PROOF: $a \iff b$

$b \iff c$

need to show $d \iff a$ $\left\{ \begin{array}{l} \text{assume (a) then show (d)} \\ \text{assume (d) then show (a)} \end{array} \right.$

Assume (a): For any $\vec{b}$, $A\vec{x} = \vec{b}$ has a solution

Suppose the solution is $\vec{x}^* = \begin{bmatrix} \vec{x}_1^* \\ \vdots \\ \vec{x}_n^* \end{bmatrix}$

Then $\left[A \middle| \vec{b} \right] \sim \left[\begin{array}{ccc|c} 1 & \cdots & 0 & x_1^* \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & x_n^* \end{array} \right] \longleftarrow$ every column of A is a pivot column.

Assume (d): $\left[A \middle| \vec{b} \right] \sim \left[\begin{array}{ccc|c} 1 & \cdots & 0 & \\ \vdots & \ddots & \vdots & \vec{d} \\ 0 & \cdots & 1 & \end{array} \right]$ and $A\vec{d} = \vec{b}$

Since nothing was assumed about $\vec{b}$, $A\vec{x} = \vec{b}$ has a solution for all $\vec{b}$

Row Multiplication of Matrices

$A = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 2 & 2 \\ 4 & 4 & 0 \end{bmatrix}$, $\vec{x} =$

Properties of Matrix VECTOR Multiplication

1. $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$

PROOF

let $A = [a_1 \dots a_n]$ $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

so $A(\vec{u}+\vec{v}) = [a_1 \dots a_n] \begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix}$

$\Longrightarrow (u_1+v_1)\underline{a}_1 + \dots (u_n+v_n)\underline{a}_n = \sum_{i=1}^{n} (u_i+v_i)\underline{a}_i$

$= \sum_{i=1}^{n} u_i\underline{a}_i + v_i\underline{a}_i$

$= \sum_{i=1}^{n} u_i\underline{a}_i + \sum_{i=1}^{n} v_i\underline{a}_i$

$= A\underline{u} + A\underline{v}$

QED ☺

Properties of Mtx Mult (cont)

1. $A(\underline{u}+\underline{v}) = A\underline{u} + A\underline{v}$

2. $A(c\,\underline{v}) = c(A\underline{v})$

Homogenous Linear Systems

$\longrightarrow$ when the right hand side of a Mtx is zero the system is in homogenous form

$$\underbrace{A}_{m\times n}\,\underbrace{\underline{x}}_{\underline{x}\in\mathbb{R}^n} = \underbrace{0}_{0\in\mathbb{R}^m}$$

Remark: Homogenous ALWAYS has the solution $\underline{x}=0$
Since it is always present it is called the trivial Solution

Example: $m=n=2$ System is two lines through the origin
Only Non-trivial solution when both equations are the same line

Ex: $A\underline{x}=0$

$$A = \begin{bmatrix} 2 & 2 & -1 \\ -1 & -2 & 2 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underline{0} \longrightarrow \begin{bmatrix} 2 & 2 & -1 & 0 \\ -1 & -2 & 2 & 0 \\ 3 & 2 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Free variable

Soulution
$x_1 + x_3 = 0$
$x_2 - \frac{3}{2}x_3 = 0$
x_3 is free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 3/2 \\ 1 \end{bmatrix}$$

Ex: $A\underline{x} = 0$ where $A = \begin{bmatrix} 1 & 0 & -2 & 3 \\ 4 & -2 & 1 & 0 \\ 3 & -2 & 3 & -3 \end{bmatrix}$

Augmented Mtx $\longrightarrow$ $\begin{bmatrix} 1 & 0 & -2 & 3 & 0 \\ 4 & -2 & 1 & 0 & 0 \\ 3 & -2 & 3 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & -9/2 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Row Solution $= x_1 \quad -2x_3 + 3x_4$

$x_2 - {}^{4}\!/\!_2 x_3 + 6x_4$

$x_3, x_4 \rightarrow$ free

$x_3 \begin{bmatrix} 2 \\ 9/2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ -6 \\ 0 \\ 1 \end{bmatrix}$

Linear Independence:

if the equation $x_1\underline{v}_1 + x_2\underline{v}_2 \cdots x_n\underline{v}_n$ has only the trivial solution. $A\underline{x}=0$

Linear Dependence:

if you can find numbers $c_1, \ldots c_p$ s.t. $c_1\underline{v}_1 + \ldots + c_p\underline{v}_p = \underline{0}$, The $\{\underline{v}_1 \ldots \underline{v}_p\}$ is linearly Dependent.

Ex: $A = \begin{bmatrix} 0 & 1 & 3 \\ 4 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & 2 & 7 \end{bmatrix}$ $A_{aug} = \begin{bmatrix} 0 & 1 & 3 & 0 \\ 4 & -1 & 1 & 0 \\ -1 & 0 & -1 & 0 \\ 1 & 2 & 7 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\left.\begin{array}{l} x_1 + x_3 = 0 \\ x_2 + 3x_3 = 0 \\ x_3 \text{ free} \end{array}\right\} x_3 \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}$

IMPORTANT:

the columns of mtx A are lin IND $\Longleftrightarrow$ $A\underline{x}=0$ has only the trivial solution

DEFINITION:

Null Space $\rightarrow$ for an $m \times n$ mtx A, $x \in \mathbb{R}$ in in the Null space of $A \Leftrightarrow A\underline{x} = \underline{0}$

THEORUM:

two non-zero vectors are DEP $\Longleftrightarrow$ they are multiples of each other.

NOTE:

DEP

two vectors which are $\boxed{\text{mult. of each other}}$ then they must lie on the same line through the origin.

→ A vector set $a_1 \dots a_n$ is Linearly IND $\Longleftrightarrow$ $A\underline{x}=\underline{0}$ has only the trivia Solution $\underline{x}=\underline{0}$

THEORUM: #7 in text

- Consider the set of vectors $S=\{\underline{v}_1 \dots \underline{v}_2\}$ for $p>2$, S is LIN DEP $\Longleftrightarrow$ at least one element of S is a linear combination of the others.

PROOF on test!!!

① Assum S is DEP and show at least ONE vector is a combination of the others

② Suppose one vector in S is a combination of the others, then show S is linearly dependent.

$\underline{v}_j$ is a linear combination of the other vectors of S, then $\underline{v}_j = c_1\underline{v}_1 + \dots + c_j\underline{v}_j + \dots + c_p\underline{v}_p$

$\Rightarrow \underline{0}$

For General: if a vector set has more elements than there are entries in each vector, then the set is dependant.

So, $S=\{\underline{v}_1, \dots \underline{v}_p\} \in \mathbb{R}^n$ is dependent if $p>n$

If A is an $m \times n$ mtx and has $\overset{\text{rows}}{m} < \overset{\text{columns}}{n}$, i.e # equation < # unknowns, then the system has AT LEAST ONE free var. so the columns are dependent.

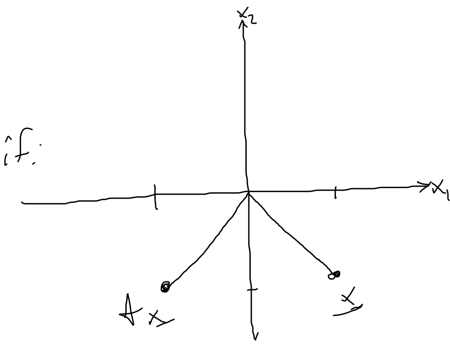
<u>Linear Transformations</u>: Consider $T(\underline{x}) = A\underline{x}$

What does A do to $\underline{x}$?

If A is $m \times n$ then $\underline{x} \in \mathbb{R}^n$, $A\underline{x} \in \mathbb{R}^m \Rightarrow T: \mathbb{R}^n \to \mathbb{R}^m$

<u>Definition</u>: The transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ has domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$

Ex $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$



<u>DEFINITION</u> $T: \mathbb{R}^n \to \mathbb{R}^m$ is a lin. transformation if:

EXAM

(1) $T(\underline{u} + \underline{v}) = T(\underline{u}) + T(\underline{v})$, $\quad \underline{u}, \underline{v} \in \mathbb{R}^n$

(2) $T(c\underline{u}) = cT(\underline{u})$, $\quad \underline{u} \in \mathbb{R}^n$, $c \in \mathbb{R}$

All mtx transformations are linear.

<u>Question</u>

(1) $T: \mathbb{R}^n \to \mathbb{R}^m$ Prove that if $\{\underline{v}_1, \ldots, \underline{v}_p\}$ is a dependent set, then $\{T(\underline{v}_1), T(\underline{v}_2), \ldots T(\underline{v}_p)\}$ is a dependent set

THINK: Linear Combinations

(2) Suppose $\{\underline{v}_1, \ldots, \underline{v}_p\}$ is an independent set. is $\{T(\underline{v}_1), \ldots, T(\underline{v}_2)\}$ an Independent set? False $\to$ find counter example

ON TEST 4

1.9 THE MTX OF A LINEAR TRANSFORMATION

DEFINITION:

The standard basis of $\mathbb{R}^n$ is the set $\{\underline{e}_1, \ldots \underline{e}_2\} \in \mathbb{R}^n$, where $\underline{e}_j$ is a vector with all zeros excep 1 in the j^{th} entry.

$\longrightarrow$ A BASIS FOR A SET is a collection of vectors which span it.

$$T(\underline{x}) = T(\underline{x}_1 \underline{e}_1 + \ldots + \underline{x}_n \underline{e}_n)$$

$$= [x_1 T(e_1) + \ldots + x_n T(e_n)]$$

$$= [T(e_1), \ldots T(e_n)] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= A\underline{x} \quad \text{where } A = T(\underline{e})$$

Theorum: there is a unique A such that $T(x) = Ax \mid \forall x \in \mathbb{R}^n$, A is an $m \times n$ mtx whose j^{th} col is $T(\underline{e}_j)$

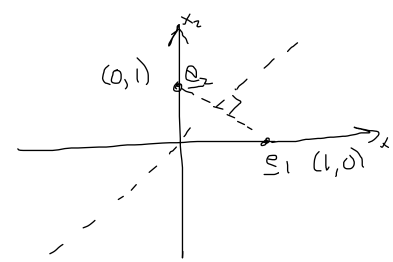
EX

$T: \mathbb{R}^2 \to \mathbb{R}^2$ T Rotates each point θ radians about the origin. Find the standard mtx of T

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Example Find mtx of Reflection through line $x_2 = x_1$

$$\left.\begin{matrix} T(\underline{e}_1) = e_2 \\ T(\underline{e}_2) = \underline{e}_1 \end{matrix}\right\} \text{standard mtx} \rightarrow A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



DEFINITION: $T: \mathbb{R}^n \to \mathbb{R}^m$ is onto $\mathbb{R}^m$ if each $\underline{b} \in \mathbb{R}^m$ is the image of at least one $\underline{x} \in \mathbb{R}^n$ (sujective)

$T: \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each $\underline{b} \in \mathbb{R}^m$ is the image of at most one $\underline{x} \in \mathbb{R}^n$ (Injective)

3/1

$A\underline{x} = \underline{b} \longrightarrow$ solve this linear system for one $n \times n$ A but many $\underline{b} \in \mathbb{R}^n$

Typically, in Real world problems A is Sparse, i.e. it contains many zeros

LU DECOMPOSITION: can we find matricies L & U s.t. L is lower diagonal $\begin{bmatrix} \boxtimes & & 0\text{'s} & \\ x & \boxtimes & & \\ x & x & \boxtimes & \\ x & x & x & \boxtimes \end{bmatrix}$

U is upper diagonal $\begin{bmatrix} \boxtimes & x & x & x \\ & \boxtimes & x & x \\ & & \ddots & x \\ 0\text{'s} & & & \boxtimes \end{bmatrix}$

Solve $A\underline{x} = \underline{b}$

$LU\underline{x} = \underline{b}$

$L(U\underline{x}) = \underline{b}$

Let $U\underline{x} = \underline{y}$, then $L\underline{y} = \underline{b} \longrightarrow L\underline{y} = \underline{b}$ has the form $\left[\begin{array}{ccc|c} 1 & & 0\text{'s} & b_1 \\ & \ddots & & b_2 \\ & & & \vdots \\ *\text{'s} & & 1 & b_n \end{array}\right]$ and is solved using substitution.

Ex: $A = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 2 & 2 \\ 0 & -1 & 2 \end{bmatrix} \longrightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 3 & 4 \\ 0 & -1 & -2 \\ 0 & 0 & 4 \end{bmatrix}$

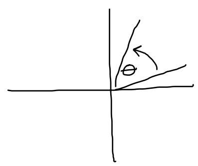
$\underline{b} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

$L\underline{y} = \underline{b}$: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 3 \end{array}\right]$ $y_1 = 2$, $y_2 = -1 - y = -3$, $y_3 = 3 - y_2 = 6$

$U\underline{x} = \underline{y}$: $\left[\begin{array}{ccc|c} 1 & 3 & 4 & 2 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 4 & 6 \end{array}\right]$ $x_3 = 2 - 3x_2 - 4x_3 = -4$, $x_2 = 3 - 2x_3 = 0$, $x_3 = 3/2$

COMPUTER GRAPHIX :

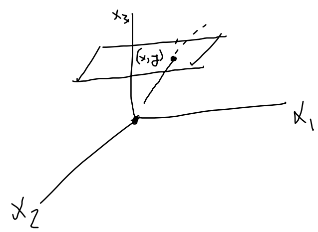
Recall that rotation about the origin & reflection are linear transforms

e.g. rotation  $m = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Translation: $(x, y) \longrightarrow (x +$

Translation is <u>NOT</u> a linear transformation

$\longrightarrow$ Homogenous coordinates.



§ 3 DETERMINANTS

2x2 mtx $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, when A is inverteble

for invetubility $A\underline{x} = \underline{0}$ has only the trivial solution

$A \sim \begin{bmatrix} a & b \\ 0 & (ad-bc) \end{bmatrix}$

$\searrow$ det(A)

For 3x3 mtx

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$$

assume $a_{11}\neq 0$ & A^{-1} is invertable

$R_2 \to a_{11}R_2 - a_{21}R_1$

$R_3 \to a_{11}R_3 - a_{31}R_1$

$$A \sim \begin{bmatrix} a_{11} & a_{12} & a_{13}\\ 0 & (a_{11}a_{22}-a_{21}a_{12}) & (a_{11}a_{23}-a_{21}a_{13})\\ 0 & (a_{11}a_{32}-a_{31}a_{12}) & (a_{11}a_{33}-a_{31}a_{13})\end{bmatrix}$$

$R_3 \to (a_{11}a_{22}-a_{21}a_{12})$

$$A \sim \begin{bmatrix} a_{11} & a_{12} & a_{13}\\ 0 & (a_{11}a_{22}-a_{21}a_{12}) & (a_{11}a_{23}-a_{21}a_{13})\\ 0 & 0 & \Delta\end{bmatrix}$$

$$\Delta = (a_{11}a_{22}-a_{21}a_{12})(a_{11}a_{33}-a_{31}a_{13})-(a_{11}a_{32}-a_{31}a_{12})(a_{11}a_{23}-a_{21}a_{13})$$

$$= a_{11}a_{22}a_{11}a_{33}-a_{11}a_{22}a_{31}a_{13}-a_{21}a_{12}a_{11}a_{33}+\boxed{\cancel{a_{21}a_{12}a_{31}a_{13}}}-(a_{11}a_{32}a_{11}a_{23}-a_{11}a_{32}a_{21}a_{13}-a_{31}a_{12}a_{11}a_{23}+\boxed{\cancel{a_{31}a_{12}a_{21}a_{13}}})$$

$$=a_{11}\left[a_{11}(a_{22}a_{33}-a_{23}a_{32})-a_{12}(a_{21}a_{33}-a_{31}a_{23})+a_{13}(a_{32}a_{21}-a_{22}a_{31})\right]$$

$$=a_{11}\left[a_{11}\det\begin{vmatrix} a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix} - a_{12}\det\begin{vmatrix} a_{21} & a_{23}\\ a_{31} & a_{33}\end{vmatrix} + a_{13}\det\begin{vmatrix} a_{21} & a_{22}\\ a_{31} & a_{32}\end{vmatrix}\right]$$

$$=a_{11}\left[a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13})\right]$$

$$A_{11}=\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix} \qquad A_{12}\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix} \qquad A_{13}\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$$

DEFINITION: (Determinant)

for $n \geq 2$ & $A = [a_{ij}]$ is $n \times n$, then $\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13}) + \ldots a_{1n}(-1)^{n+1}\det(A_{1n})$

Notation: $|A| = \det(A)$

COFACTOR: Given $n \times n$ Mtx $A = [a_{ij}]$, the i^{th}, j^{th} cofactor

$$C_{ij} = (-1)^{i+j}\det(A_{ij})$$

Theorem: Det (A) can be found by expanding along the i^{th} row

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in} = \sum_{k=1}^{n} a_{ik}C_{ik}$$

$$\left.\begin{bmatrix} + & - & + & - & + & - & \cdots \\ - & + & - & + & & & \\ + & - & + & - & & & \\ - & + & - & + & & & \\ + & & & & \ddots & & \\ \vdots & & & & & & \ddots \end{bmatrix}\right\} \text{signs of the Cofactor matrix}$$

the determinant of an upper echelon or lower echelon matrix is the product of its diagonal entries

Some row operations change the sign of the determinant:

1. row exchange - changes sign

2. addition of the multiple of one determinant to another - does not change the determinant

3. the scalar multiple of rows - multiplies the determininant by the same multiple.

THEORUM 3- **A = [a(ij)]**

1. if a multiple of one row of A is adeed to another to make a new matrix B then det(A)=det(B)

2. If two rows of A are interchanged to make new matrix B then det(A)=-det(B)

3. If a matrix is formed by taking a row of A & multiplying by a faxtor of r, then det(A)*r=det(B)

Ex: $A = \begin{bmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & -2 & 3 \end{bmatrix}$ find det(A)

$$\det(A) = \begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & -2 & 3 \end{vmatrix} \begin{matrix} R_3 \to R_3 + R_1 \\ R_4 \to R_4 - 3R_1 \end{matrix} \begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 1 & 5 & 5 \\ 0 & 2 & 7 & 3 \end{vmatrix} \begin{matrix} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - 2R_2 \end{matrix} \begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -3 & -5 \end{vmatrix} \; R_4 \to R_3 \; - \begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -3 & -5 \end{vmatrix}$$

Row interchange = Sign change

$= 1 \times 1 \times -3 \times 1 = 3$

PROPERTIES OF THE DETERMINANT

1. $\det(AB) = \det(A)\cdot\det(B) = \det(BA)$ → if $\det(AB)=0$, then either A or B or both are not invertable.
   → Conversly, if $\det(AB)\neq 0$ the BOTH A & B must be invertable.
2. $\det(A^T) = \det(A)$
3. $\det(A+B) \neq \det(A)+\det(B)$
4. $\det(A^{-1}) = \frac{1}{\det(A)}$

Ex:

Suppose A is nilpotent, i.e. $\exists k \mid A^k = 0$ Then show A is not invertable. $(\det(A)=0)$

$\det(A^k) = 0$

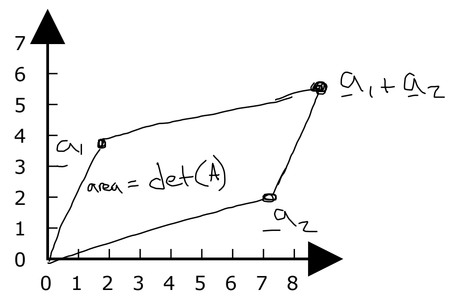
$\det(A^k) = \det(A)^k$,

$\therefore$ Since $k>0$, $\det(A)=0$

DETERMINANTS AS AREAS AND VOLUMES

If A is a 2x2 matrix, then $\det(A)$ is the Area of the parallelogram determined by the columns of A.

$A = [\underline{a}_1, \underline{a}_2]$



In $\mathbb{R}^3$, if A is 3×3 matrix then $|\det A|$ is the volume of the parallelepiped determined by the colums of A.

<u>Interpetation of $\det A = 0$</u>: If the columns of A are dependent, then A must have a determinant zero, i.e. the $\frac{\text{volume } \mathbb{R}^3}{\text{area } \mathbb{R}^2}$ of the region must be 0
then A is <u>NOT</u> invertable